Generalized Weierstrass representation for surfaces and Lax-Phillips scattering theory for automorphic functions

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Abstract

Relation between generalized Weierstrass representation for conformal immersion of generic surfaces into three-dimensional space and Lax-Phillips scattering theory for automorphic functions is considered.

It is well-known that Poincare plane Π, i.e., the upperhalf plane

$$y > 0, \quad -\infty < x < \infty, \quad z = x + iy$$

be the model of Lobachevsky geometry, where the role of motion group played the group $G = SL(2, \mathbf{R})$ of fractional linear transformations

$$z \longrightarrow zg = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where $a, b, c, d \in \mathbf{R}$, $ad - bc = 1$.

The group $SL(2, \mathbf{R})$ has a great number of so-called *discrete subgroups*. The sugroup Γ is called discrete if the identical transformation is isolate from the other transformations $\gamma \in \Gamma$. For example, *a modular group* consisting of transformations with integer a, b, c, d is discrete subgroup. Further, *a fundamental domain* F of discrete subgroup Γ be an any domain on Poincare plane such that the every point of Π may be transfered into a closing $\bar{F}$ of domain F by means of some transformation $\gamma \in \Gamma$, at the same time no

there exists the point from F which transfered to the other point of F by such transformation. The function f defined on Π is called *automorphic* with reference to discrete subgroup Γ if

$$f(\gamma z) = f(z), \quad \gamma \in \Gamma.$$

Further, generalized Weierstrass representation for surfaces was proposed by Konopelchenko in 1993 [1, 2] is defined by the following formulae

$$\begin{aligned} X^1 \quad + \quad iX^2 &= i\int_\epsilon (\bar{\psi}^2 dz' - \bar{\varphi}^2 d\bar{z}'), \\ X^1 \quad - \quad iX^2 &= i\int_\epsilon (\varphi^2 dz' - \psi^2 d\bar{z}'), \\ X^3 \quad &= \quad -\int_\epsilon (\psi\bar{\varphi} dz' + \varphi\bar{\psi} d\bar{z}'), \end{aligned} \tag{2}$$

where ϵ is arbitrary curve in $\mathbf{C}$, ψ and φ are complex-valued functions on variables $z, \bar{z} \in \mathbf{C}$ satisfying to the linear system (two-dimensional Dirac equation):

$$\begin{aligned} \psi_z &= U\varphi, \\ \varphi_{\bar{z}} &= -U\psi, \end{aligned} \tag{3}$$

where $U(z, \bar{z})$ is a real-valued function. If to interpret the functions $X^i(z, \bar{z})$ as coordinates in a space $\mathbf{R}^{3,0}$, then the formulae (2), (3) define a conformal immersion of surface into $\mathbf{R}^{3,0}$ with induced metric

$$ds^2 = D(z, \bar{z})^2 dz d\bar{z}, \quad D(z, \bar{z}) = |\psi(z, \bar{z})|^2 + |\varphi(z, \bar{z})|^2,$$

at this the Gaussian and mean curvature are

$$K = -\frac{4}{D^2}[\log D]_{z\bar{z}}, \quad H = \frac{2U}{D}. \tag{4}$$

Let us consider a closed surface with genus > 1, and let $F : \Sigma \longrightarrow \mathbf{R}^{3,0}$ be an immersion of surface with genus > 1 given by (2)-(3). It is well-known that every closed oriented surface Σ with positive genus is uniformizable:

$$\rho : M \longrightarrow \Sigma,$$

where a surface M is conformal covering. Hence it immediately follows that a factor-space M/Γ is conformally equivalent to the surface Σ, where Γ is a

discrete subgroup of a group of isometries of M. In our case a space M is isometric to the Poincare plane Π. The group of isometries of Π is the group $G = SL(2, \mathbf{R})$, the transformations of which are defined by (1).

According to [3] (Proposition 4) we have that a surface Σ with genus > 1 immersing into $\mathbf{R}^{3,0}$ by formulas (2)-(3) is conformally equivalent to a surface Π/Γ, where Γ is a discrete subgroup of $SL(2, \mathbf{R})$. The functions ψ and φ, the metric tensor $D(z)^2$ and potential $U(z)$, are transformed by elements of Γ as follows

$$\begin{array}{rcl} \psi(\gamma(z)) & = & (c\bar{z} + d)\psi(z), \\ \varphi(\gamma(z)) & = & (cz + d)\varphi(z), \\ D(\gamma(z)) & = & |cz + d|^2 D(z), \\ U(\gamma(z)) & = & |cz + d|^2 U(z). \end{array}$$

Hence it immediately follows from (4) that

$$H(\gamma(z)) = H(z), \quad \gamma \in \Gamma. \tag{5}$$

Therefore, *the mean curvature is automorphic function.*

Further, follows to [4] let us consider the discrete subgroup $\Gamma \subset SL(2, \mathbf{R})$ satisfying to the following requirements:

1. A space $SL(2, \mathbf{R})/\Gamma$ is noncompact.

2. Γ contains the only one parabolic subgroup.

The fundamental domain $F_\Gamma = F$ for the group Γ choosing as follows

a. F lies on a strip $-X < x < X,\ y > Y > 0$.

b. Intersection $F \cap \{y > d\}$ at the some $d,\ d > 1$, is coincide with a strip $-X_1 < x < X_1,\ y > d$.

c. A boundary of F is smooth and consists of geodesic segments with a finite number of corner points.

In Hilbert space $L_2(F, d\mu)$, where $d\mu = y^{-2}dxdy$ is a measure, consider a symmetric operator L defined by the differential expression

$$L = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{1}{4} \tag{6}$$

on all sufficiently smooth and uniformly restricted in F the functions H, which satisfying to automorphity condition (5). A spectrum of the operator L consists of finite set of own values: $\lambda_0 = -\frac{1}{4}$ (this own number corresponds to the unit representation of group $SL(2, \mathbf{R})$), the numbers $\lambda_l = -\mu_l^2,\ l = 1, 2, \ldots, N$ are belong to $(-\frac{1}{4}, 0)$ (additional series), the set of positive own values $\lambda_l,\ l = N+1, N+2, \ldots,\ \lambda_l \in (0, \infty)$ (basic series), and the branch of absolutely continuous spectrum $\lambda = k^2$ on $[0, \infty]$.

If we assume that functions H compose the basis of Hilbert space $L_2(F, d\mu)$, then *the automorphic wave equation* may be written as

$$H_{tt} + LH = 0, \tag{7}$$

where operator L has the form (6). This equation naturally defines a group $\mathcal{V}_t$ of transformations (smooth and finite) of Cauchy data $\mathcal{U}(z,t) = \begin{pmatrix} u(z,t) \\ \frac{\partial}{\partial t} u(z,t) \end{pmatrix}$, the action of which expressed by the formula

$$\mathcal{U}(z,t) = \mathcal{V}_t \mathcal{U}(z,0).$$

The group $\mathcal{V}_t$ has orthogonal in- and out-spaces $\mathcal{D}_-,\ \mathcal{D}_+$ are satisfying to conditions

$$\begin{array}{ll} 1)_- & \mathcal{V}_t \mathcal{D}_- \subset \mathcal{D}_-,\ t < 0, \\ 1)_+ & \mathcal{V}_t \mathcal{D}_+ \subset \mathcal{D}_+,\ t > 0, \\ 2) & \bigcap\limits_{t<0} \mathcal{V}_t \mathcal{D}_- = \bigcap\limits_{t>0} \mathcal{V}_t \mathcal{D}_+ = 0, \\ 3) & \bigcup\limits_{t>0} \mathcal{V}_t \mathcal{D}_- = \bigcup\limits_{t>0} \mathcal{V}_t \mathcal{D}_+, \\ 4) & \mathcal{D}_- \perp \mathcal{D}_+. \end{array}$$

These conditions allow to apply the Lax-Phillips framework [5] and to find generalized own functions $e(z)$ of automorphic wave equation (7) which are expressed via the Eisenstein series, and also to define the spectrum representation of operator L and scattering matrix.

References

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